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On the development of Mach waves radiated by small disturbances

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Mach wave radiation studies have, so far, been concentrated on the sound radiated to large distances from the flow. Then, both a turbulent eddy and the distance it may travel during its coherent life, appear small to the distant observer, so that the sound arrives from one direction. When that direction is the Mach angle, a Mach wave is heard. This paper deals with a different situation, where, although a turbulent eddy appears small, the distance it travels does not. Sound arriving at the observer then comes from different directions at different times in the eddy's life, so that Mach waves can only be radiated over a relatively small range, where the radiation angle corresponds to the Mach angle. In that range the far field equations no longer apply. It is shown that, whereas the distant field increases with the cube of convection Mach number, M, and inversely with the square of distance travelled, $1/r^2$, this particular near field is of a type where the mean square density, $\overline{\rho^2}$, has the proportionality

$$\overline{\rho^2} \sim \bar{\rho}^2 \frac{l^{\frac{3}{2}}}{r^{\frac{3}{2}}} \frac{M^{\frac{7}{2}}}{(M^2-1)^{\frac{1}{2}}}.$$

1. Some experimental findings

The theory of aerodynamic sound production by supersonic turbulent flows throws emphasis on the highly directional Mach wave radiation. The theory predicts many features which are in accord with experimental data (Lighthill 1963) and this agreement has given emphasis to the suggestion that Mach waves are the dominant source of noise in rocket exhaust flows. Very little is understood about the turbulence in highly sheared supersonic flow, and the Mach waves it emits have not as yet been subjected to detailed experimental study, although the work reported by Laufer (1961, 1962, 1964) is a noted exception. Analytically, the connection between the Mach waves and the turbulent sources is an extremely simple one, an aspect that suggests that experiments carried out in the near field of rocket flows can be interpreted to throw light on the turbulence problem. Early shadowgraph pictures of Mach waves emitted by supersonic flow (such as those obtained at both the Langley Research Center in the U.S.A. and by Ricketson‡ in England) show only straight wave-fronts that may be regarded as sections of Mach cones generated by the

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rapidly moving eddies. However, more recent pictures taken at the Langley Research Center have resolved waves in a much later stage of development, and indicated what is apparently a strong system of spherical wavefronts that form some 20 or so nozzle diameters from the flow. Although it was always clear that the small straight wave-fronts could not be self-preserving, and would eventually develop into spherical fronts, that aspect of the problem had been passed over very lightly in the early theories. But the clear evidence of spherical wave formation evident in the recent pictures brings forward this point very forcibly. Important questions are raised, not the least of which is, whether or not the intense spherical waves are in fact developed from the elementary Mach waves visible in the pictures nearer the turbulent flow? Even if they are, that development will have to be understood before one can gain much value from near field studies. It was an attempt to answer some of these questions that prompted the present analytical study of Mach wave development. It soon became evident that there was very little subtlety in the non-linear growth of distant spherical waves, for they should be nothing more than the spherical waves whose coalescence near the flow, formed the conical Mach fronts. Obviously, they will be centred on their position of origin, a point that can easily be checked from the shadowgraph pictures. The study did, however, raise a more interesting point regarding the existence of a near Mach wave-field. This is of course contrary to our initial conception of the problem where simple sources, which have no near field, are the radiators. But the near field is of rather a novel kind and it is that aspect we describe below.

2. The theory of quadrupole Mach wave emission

The general aerodynamic noise equation relates the radiated density to a simple source distribution in the turbulent flow

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^2}\int \frac{\partial^2 T_{ij}}{\partial y_i\,\partial y_j}\left(\mathbf{y}, t-\frac{r}{a_0}\right)\frac{d\mathbf{y}}{r}. \tag{2.1}$$

The simple source strength, being a double divergence, integrates instantaneously to zero, showing that the sources are arranged throughout the flow in opposing pairs so that they form multipoles. Lighthill showed how they are equivalent to a volume distribution of quadrupoles, and how this property requires a modification of equation (2.1) that displays the quadrupole features more directly. When in the far field of each individual correlated region, the proper alternative form may be written

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^4 r}\int \frac{\partial^2 T_r}{\partial t^2}\left(\mathbf{y}, t-\frac{r}{a_0}\right)d\mathbf{y}, \tag{2.2}$$

where

$$T_r = (\partial r/\partial y_i)\,(\partial r/\partial y_j)\,T_{ij}. \tag{2.3}$$

Convection of the turbulence with constant speed $a_0 M$ in the downstream, or 1, direction gives the tensor time derivative $\partial^2 T_r/\partial t^2$ a deceptively large appearance, for it consists, in part, of terms of the type $a_0 M(\partial T_r/\partial y_1)$. Such terms are themselves space derivatives, or divergences, which, for the same reason that prompted the development of equation (2.2) from equation (2.1), integrate to zero and generate no sound. To overcome this difficulty and to emphasize that small part of the turbulence responsible for the generation of sound, Lighthill chose moving axes, co-ordinates $\boldsymbol{\eta}$, and showed how†

$$\frac{\partial^2 T_r}{\partial t^2}\left(\mathbf{y}, t-\frac{r}{a_0}\right) \int = \frac{\partial^2 T_r}{\partial t^2}\left(\boldsymbol{\eta}, t-\frac{r}{a_0}\right)(1-M\cos\theta)^{-2}, \tag{2.4}$$

where θ is the angle of sound emission measured from the downstream direction. The volume element was also changed in the moving-axis system, but for the present we shall not introduce that effect here, since it follows naturally in the work we describe.

The most refined equation describing sound emission by convected turbulence is then

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^4 r}\int \frac{1}{(1-M\cos\theta)^2}\frac{\partial^2 T_r}{\partial t^2}\left(\boldsymbol{\eta}, t-\frac{r}{a_0}\right)d\mathbf{y}. \tag{2.5}$$

We use the term 'refined' deliberately, for this equation is in essence no more correct than any of the others; they are all exact in the far field. The developments are essentially of a type that bring into the open important features of the radiation and do not leave them hidden in some subtle property of the integral.

It is evident that equation (2.5) is singular, whenever $(1-M\cos\theta) = 0$. We know now that this singularity is associated with Mach waves, which result when each simple source radiates independently of its opposing partner so that the quadrupole equation is no longer the one that best describes it. We must then revert to the simple source equation (2.1) expressed in the more convenient Mach wave form (Ffowcs Williams 1963)

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^2 r}\int \frac{\partial^2 T_{rr}}{\partial y_r^2}\left(\mathbf{y}, t-\frac{r}{a_0}\right)d\mathbf{y}. \tag{2.6}$$

Here again, the suffix r implies the radiation direction, and repeated suffices are not to be summed. The expression of the two equations (2.5) and (2.6) in a form that best describes the sound of a single eddy requires one further step. The total radiation will depend on the volume occupied by the eddy during its radiation, or retarded, time. This is clearly true since the equation integrates a constant strength density only over that volume where a source exists at the proper time. If the motion of the source is always such as to place it in a position where it can emit irrespective of time, i.e. if it approaches the observer with precisely the speed of sound, then it will appear that an infinite volume would contain a finite source, and the sound would be arbitrarily large. This is really the case where Mach waves are relevant, for the more distant parts of the eddy emit first. At a later time the nearer elements make their contribution, but by then the distant part already considered has moved to the new location and is to be considered again. The singular emission is avoided by recognizing that the eddy has a limited lifetime, τ say, so that it only exists over a length $a_0\tau$ in the radiation direction. The Mach wave equation can then be approximated by

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0 r} l^2\tau \frac{\partial^2 T_{rr}}{\partial y_r^2}\left(\mathbf{y}, t-\frac{r}{a_0}\right). \tag{2.7}$$

† The symbol $\int =$ implies that related functions generate identical distant sound fields; i.e. they integrate at retarded time to identical values.

This is to be interpreted as the Mach wave of a single eddy. The volume occupied by the quadrupole during emission in one of its more classical phases is well known to be $l^3(1-M\cos\theta)^{-1}$ (Lighthill 1952, Ffowcs Williams 1963) so that equation (2.5) can be approximated by

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^4 r}\frac{l^3}{(1-M\cos\theta)^3}\frac{\partial^2 T_r}{\partial t^2}\left(\mathbf{y}, t-\frac{r}{a_0}\right). \tag{2.8}$$

This represents the quadrupole radiation from a single eddy. These equations are the ones familiar to the theory and will now be used to generate the description of a transition field, where although the distance r may put the observer in the far field of an eddy at any one instant, he is still close in comparison with the distance travelled by the eddy during its coherent life.

3. Near field Mach wave emission

We have seen that an essential feature determining the strength of Mach waves radiating to large distances is that they emit coherently over their entire lifetime so that they occupy an effective volume $a_0 \tau l^2$. It is clearly a necessary condition for the analytical consistency of the theory that all that volume should be contained in the narrow region where the angle made between the observation point and the direction of flow should be $\theta = \cos^{-1} M^{-1}$. If that were not so, the eddy

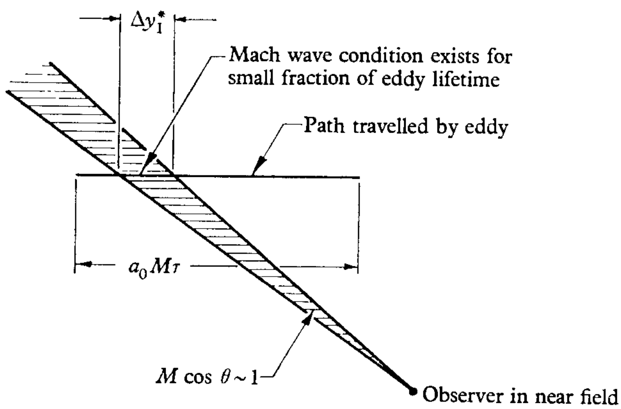

FIGURE 1. Diagram illustrating the finite region from which an eddy emits Mach waves to the near field.

would only remain in its Mach wave phase for a fraction of its lifetime, for θ would vary singificantly (see figure 1). It is the modification of the theory to account for this effect that we shall describe. It has been carried out with the aim of extending our understanding of the high Mach number noise theory into the near field regions.

The extension we seek follows directly once we can specify the effective volume occupied by an eddy during the time interval, where sound that arrives at the particular observation point $(\mathbf{x}, t)$ is being generated.

We assume that the eddy is of characteristic length l and that it moves downstream with constant speed $a_0 M$. A fixed point on the eddy would then trace out a path, $y_1 = \eta_1 + a_0 Mt$, where η is the streamwise position of that point at

$t = 0$; i.e. the fixed and moving co-ordinate system, $\mathbf{y}$ and $\boldsymbol{\eta}$, respectively, coincide at $t = 0$.

Its downstream position at the time when it must emit sound to reach $(\mathbf{x}, t)$ will then be the particular value of $\eta_1 + a_0 Mt$ at the retarded time, $t - r/a_0$, y_1^* say:

$$y_1^* = \eta_1 + a_0 Mt - Mr = \eta_1 + a_0 Mt - M[(x_1 - y_1^*)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}, \tag{3.1}$$

where the co-ordinate origin in the directions 2 and 3 has been assumed in the eddy path.

Similarly, the path traced out by a moving point, an eddy distance l behind the first point will be, $y_1 + \Delta y_1 = \eta_1 + l + a_0 Mt$, so that it emits sound at position $y_1^* + \Delta y_1^*$,

$$y_1^* + \Delta y_1^* = \eta_1 + l + a_0 Mt - M[(x_1 - y_1^* - \Delta y_1^*)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}. \tag{3.2}$$

The total effective downstream distance over which an eddy emits is then Δy_1^* and this is the quantity we now seek in extending the far field theory to a near field situation. It is tedious but straightforward to solve equation (3.2) to bring out the explicit dependence of Δy_1^* on eddy size and convective effects, and we find two characteristic values. The first is the familiar result for convected sources

$$\Delta y_1^* = \frac{l}{(1 - M\cos\theta)}. \tag{3.3}$$

The second result is quite different, and is the one valid in regions where (3.3) predicts Δy_1^* to be impossibly large

$$\Delta y_1^* = \left\{\frac{2lrM}{(M^2-1)}\right\}^{\frac{1}{2}} = \text{length over which an eddy emits Mach waves.} \tag{3.4}$$

The range in the direction of emission is then

$$\left\{\frac{2lr}{M(M^2-1)}\right\}^{\frac{1}{2}}. \tag{3.5}$$

The limit on the near field régime is set when the observation point is sufficiently far from the source that the eddy remains a Mach wave generator throughout its coherent life. Then, as we have already discussed, the effective range Δy_1^* is $a_0 M\tau$.

The near field may now be defined more accurately as being within the régime where

$$a_0 \tau > \left\{\frac{2lr}{M(M^2-1)}\right\}^{\frac{1}{2}}. \tag{3.6}$$

We notice that for a given time scale, the near field becomes more extensive as the Mach number is increased, its edge, r_n say, being defined by a rearrangement of the limiting value of the inequality, equation (3.6),

$$r_n = \frac{a_0^2 \tau^2}{2l} M(M^2-1). \tag{3.7}$$

This is probably best expressed in terms of the length of the coherent waves clearly visible in shadowgraph pictures. Their length, L say, is $a_0 M\tau \sin\theta$, so that $L = a_0 \tau (M^2-1)^{\frac{1}{2}}$ and r_n, from equation (3.7), is

$$r_n = \tfrac{1}{2}(L^2/l)M. \tag{3.8}$$

It is difficult to conjecture on the relative scales involved, but one can see that positions downstream are associated with lower Mach numbers and greater turbulent lengths, both of which tend to reduce the volume where this particular type of near field is active. It is tempting to speculate that the region where the near field might be most evident, near the nozzle exist, is also the region where highly directional wavelets are visible in shadowgraph pictures, and that their well defined front is more associated with the near field properties than with the far field Mach wave emission. But it would be wrong to emphasize that point, because it is unnecessary to make any such assumption in understanding their clarity near the nozzle. The waves are in themselves focused in the Mach wave direction and can combine in coherent conical fronts only as long as they remain close to their source. They are, of course, formed by superposition of spherical waves and as soon as the observation point is sufficiently distant from the source, their spherical nature is bound to be evident. The spherical fronts will be centred on the general region of their birth, a feature that will undoubtedly prove useful in assessing the major sound producing regions of the flow.

We conclude this discussion of the near field with the main result of the present analysis. That is, that the Mach waves seem to have a region in their early development where their strength falls off with distance from source less rapidly than it does in the far field. That strength is readily predicted by making use of equation (3.4) in a dimensional estimate of equation (2.6)

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^2 r^{\frac{1}{2}}}\frac{\partial^2 T_{rr}}{\partial y_r^2}\left\{\frac{2l}{M(M^2-1)}\right\}^{\frac{1}{2}} l^2. \tag{3.9}$$

Employing the usual dimensional analysis, this result predicts the mean square density in the near field of one eddy to vary as

$$\overline{\rho^2} \sim \bar{\rho}^2 \frac{M^3}{(M^2-1)}\frac{l}{r}. \tag{3.10}$$

In the direction of emission there are $\{2r/[M(M^2-1)\,l]\}^{-\frac{1}{2}}$ eddies capable of emitting Mach waves within a length l. This is clear from equation (3.5) showing how each eddy occupies a length that is the inverse of this value. In other perpendicular directions, since the integration is then an instantaneous one, the number of eddies per unit area is independent of Mach number. This factor modifies the result in equation (3.10), to make the near field Mach wave strength, emitted by a turbulent volume l^3, proportional to

$$\overline{\rho^2} \sim \bar{\rho}^2 \frac{l^{\frac{3}{2}}}{r^{\frac{3}{2}}}\frac{M^{\frac{7}{2}}}{(M^2-1)^{\frac{1}{2}}}. \tag{3.11}$$

These values contrast sharply with the far Mach wave field where each eddy radiates a wave with the proportionality

$$\overline{\rho^2} \sim \bar{\rho}^2 M^2 l^2/r^2, \tag{3.12}$$

while the mean square density radiated in the Mach waves emitted by a fixed volume l^3 has the familiar M^3 proportionality

$$\overline{\rho^2} \sim \bar{\rho}^2 M^3 l^2/r^2. \tag{3.13}$$

From these results it would seem that the radiated field increases more rapidly with increasing Mach number than does the near field. The difference, however, is not great, being a factor proportional to $(M-M^{-1})^{\frac{1}{4}}$ for the pressure levels in the two régimes. This point is qualitatively in accord with measurements of radiated and surface pressure near a supersonic turbulent boundary layer (Laufer 1961; Kistler & Chen 1963) but that situation is not one to which the present theory is well suited. It would be more likely to relate the far field pressure with that at the outer edge of the boundary layer, or with the near and far field pressures in a rocket exhaust flow, but such measurements have not yet been attempted. Should they be contemplated, it is hoped that the theory presented herein would aid the interpretation of results and help make clear the relevance of the Mach wave field to practical flow noise problems.

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